JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **201,** 329–338 (1996)
ARTICLE NO. 0258

Existence and Uniqueness of Scattering Solutions in Non-smooth Domains

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Submitted by William F. Ames

Received October 3, 1995

A short and self-contained proof of the existence of the scattering solution in exterior domains is presented for some class of second order elliptic equations. The method does not use the integral equation; it is based on Fredholm theory and the limiting absorption principle for solutions in the whole space. It covers domains with Lipschitz boundaries, domains satisfying a cone condition, and those with the so-called local compactness property.

1. INTRODUCTION

Consider the scattering problem

$$Lu := lu + K^2u = 0 \qquad \text{in } D' \tag{1}$$

$$\Gamma u = 0 \qquad \text{on } S, \tag{2}$$

$$u = u_0 + v, \qquad u_0 := \exp(iK\alpha \cdot x),\ \alpha \in \mathbb{S}^{n-1}, \tag{3}$$

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† Second author is supported by Spanish DGYCT PB94-0192.

0022-247X/96 $18.00

where v satisfies the radiation condition

$$\lim_{r\to\infty}\int_{|x|=r}|v_r - iKv|^2 ds = 0, \tag{4}$$

$D \subset \mathbb{R}^n$, $n > 2$ is a bounded domain not necessarily connected, $D' := \mathbb{R}^n \setminus D$, $S = \partial D$, Γu is a boundary condition which is assumed to be one of

$$\Gamma u = u \qquad \text{(Dirichlet BC)}, \tag{2D}$$

$$\Gamma u = u_N \qquad \text{(Neumann BC)}, \tag{2N}$$

where u_N is the conormal derivative with respect to the principal part $\tilde{l}$ of the operator l, $\tilde{l}u := \partial_i(a_{ij}(x)\partial_j u)$, i.e., $u_N := a_{ij}u_j N_i$, $u_j = \partial u/\partial x_j$ (here and below summation is understood over the repeated indices),

$$\Gamma u = u_N + \sigma(s)u \qquad \text{(Robin BC)}. \tag{2R}$$

Here N is the unit exterior normal to S, $\sigma(s)$ is a continuous function on S, $s \in S$, $K > 0$ is a constant, and

$$lu := \partial_i\big(a_{ij}(x)\,\partial_j u\big) - q(x)u \tag{5}$$

is an elliptic formally symmetric, with real-valued coefficients, differential operator in $\mathbb{R}^n$; i.e., for every $t \in \mathbb{C}^n$ there exist possitive constants C_1, C_2 such that

$$C_1 t_j \bar{t}_j \le a_{ij}(x) t_i \bar{t}_j \le C_2 t_j \bar{t}_j, \qquad a_{ij} = a_{ji} = \bar{a}_{ij}. \tag{6}$$

We assume that

$$a_{ij} = \delta_{ij} \qquad \text{if } |x| > a,\ a_{ij} \text{ Lipschitz continuous}, \tag{7}$$

$$q = \bar{q}, \qquad q \text{ is an } L^\infty - \text{function compactly supported in the ball } |x| < a, \tag{8}$$

$$0 \le \sigma(s) \in C(S). \tag{9}$$

Conditions (7) and (8) are sufficient for the unique continuation property (Lemma 2 below) to hold. At the end of the paper in Remark 1 we discuss possible generalizations.

In the sequel we denote by B_r the ball of radius r centered at the origin, $S_r := \partial B_r$ and $D_r := D' \cap B_r$, $r > a$, and assume that $S \subset B_a$. $L[.,.] := a_{ij}(.)_i(.)_j$ denotes the bilinear form associated with the principal part of L.

The classical formulation of the problem (1)–(4) is clear if S is sufficiently smooth. If S is not smooth (rough domain) then it is necessary to define the meaning of (2).

The following assumption (A) concerning the class of domains D will be crucial:

(A1) For $r > a$ the embedding

$$i : H^1(D_r) \to L^2(D_r) \text{ is compact.}$$

(A2) The trace operator

$$r : H^1(D_r) \to L^2(S) \text{ exists and is compact.}$$

These assumptions restrict the smoothness of D implicitly. They are rather weak and are satisfied in most of the practically interesting cases: Lipschitz domains and domains with cone property are admissible, as we explain below. Assumption (A2) is needed only for the Robin condition; we do not use it for the Neumann condition, while for the Dirichlet condition no assumptions on D are needed, except boundedness of D.

Assumptions (A) hold, for instance, if D is a Lipschitz domain $D \in C^{0,1}$, that is, a domain whose boundary is locally the graph of a Lipschitz function, and the Lipschitz constant does not depend on the local patch of the boundary.

Also the following cone property for D suffices for (A) to hold: For any $p \in \overline{D}$ there exists a cone $\mathscr{K}$ (with the vertex at p) contained in $\overline{D}$ together with its closure,

$$\mathscr{K} := \left\{ x : |x'|^2 \le b x_n^2, 0 < x_n < a; b > 0, x' = (x_1, \dots, x_{n-1}) \right\}.$$

The class of domains having the cone property is larger than the class of Lipschitz domains as defined above.

Assumption (A) holds also if $D \in EV_2^1$, the class of domains for which there exists a bounded extension operator

$$E : H^1(D) \to H^1(\mathbb{R}^n), \qquad Eu(p) = u(p) \text{ if } p \in D.$$

This class also contains the Lipschitz domains.

If we deal with boundary condition (2N) then we only consider assumption (A1) on S. For the Dirichlet problem we can drop the whole assumption (A) and assume just that D is a compact domain.

In [2, p. 243] a necessary and sufficient condition for (A1) to hold is given. In [1] the boundary-value problems for second order elliptic equations in bounded Lipschitz domains are studied.

We prove in the present work, that for the class of compact domains satisfying condition (A) appropriate to the boundary condition, the solution to (1)–(4), as defined above, exists and is unique. In the proof we will use two well known results:

LEMMA 1 (*Rellich's Type Lemma, see* [3, *p*. 25]). *Let u be a solution of* $(\Delta + K^2)u = 0$ *in* $|x| > R$, *such that* $\int_{|x|=r} |u|^2 ds \to 0$ *as* $r \to \infty$. *Then* $u = 0$ *for* $|x| > r$.

LEMMA 2 (Unique Continuation). *Solutions of* (1) *which vanish on an open subset of* D' *must vanish everywhere in* D'.

In Section 2 we give the definition of solution for non-smooth domains and we prove the uniqueness result. In Section 3 we prove the existence of solution to (1)–(4) and discuss generalizations.

For smooth domains integral equations methods were widely used for solutions of (1)–(4) (see [3] and references therein). For non-smooth domains for Laplace's operator and Neumann boundary condition, one can find in [5] a review of the limiting absorption principle for proving the existence of the solutions to (1)–(4). In this paper we emphasize the method of proof which is based on the Fredholmness of our problem and does not use boundary integral equations. The method allows us to handle fairly rough domains with minimal technical difficulties. Throughout the paper we assume that the domains satisfy assumption (A) and will not repeat this assumption. In [4] uniqueness of the solution to the inverse obstacle scattering problem for non-smooth boundaries is proved.

2. UNIQUENESS

Let us start with the definition of the solution for general domains.

DEFINITION 1. We say that u solves the scattering problem if, for every $r > a$,

(a) $u \in \mathscr{H}$,

(b) for all test functions $w \in H^1(D')$, vanishing near infinity, the following integral identity holds,

$$\int_{D'} \left(K^2 u\overline{w} - qu\overline{w} - a_{ij} u_j \overline{w}_i \right) dx + \int_S \sigma u\overline{w} ds = 0, \tag{10}$$

(c) u satisfies (3) and (4).

If Neumann (2N) or Robin condition (2R) are imposed then $\mathscr{H} = H^1(D_r)$. For the Neumann boundary condition the boundary integral in (10) is dropped. If the Dirichlet condition (2D) is imposed then $\mathscr{H} = \mathring{H}^1(D_r)$ ($\mathring{H}^1(D_r)$ is the space of functions in $H^1(D_r)$ vanishing on S), the boundary integral in (10) is dropped, and the test function w has to be taken in $\mathring{H}^1(D')$.

Condition (4) makes sense: from (7), (8), and the regularity results for the weak solutions of (1), it follows that v is smooth in D'_a. Clearly if D is a Lipschitz domain, then (10) implies Eq. (1) and the boundary condition (2). If $\sigma = 0$ then the last integral in (10) is absent. The easier case of the Dirichlet condition is left to the reader as an exercise. In this case, the last integral in (10) vanishes and w runs through the subspace of $H^1(D'_r)$ of functions vanishing on S in the sense of the embedding theorem.

Let us state the main result of this section.

THEOREM 1. *If D is compact and assumptions* (6)–(9) *hold, then the solution to the scattering problem is unique.*

Proof. We first pass from the homogeneous equation (1) to the equivalent non-homogeneous one in order to eliminate u_0 from the asymptotic condition at infinity (3) and (4). This passage is standard. Fix $r_0 > a$, take $h \in C^\infty(\mathbb{R})$, such that $h'(r) \geq 0$, $h(r) = 1$ if $r > r_0 + 1$, $h(r) = 0$ if $r < r_0$, and define $W(x) = v(x) + (1 - h(|x|))u_0(x)$. Inserting $u = W + hu_0$ in (10), using (8), (9), and Green's formula, we obtain

$$\int_{D'} \left(K^2 W\overline{w} - qW\overline{w} - a_{ij}W_j\overline{w}_i + f\overline{w} \right) dx + \int_S \sigma W\overline{w} = 0, \tag{11}$$

where $f := (\Delta + K^2)(hu_0)$ is a C^∞ function supported in $r_0 < |x| < r_0 + 1$.

Since $W(x) = v(x)$ if $|x| > r_0 + 1$, W satisfied the radiation condition

$$\lim_{r\to\infty} \int_{|x|=r} |W_r - iKW|^2 ds = 0. \tag{12}$$

To prove the uniqueness of the solution it is enough to check that from (11) with $f = 0$, and (12), it follows that $W = 0$. To do this, we use the standard strategy [3].

From (12) we obtain

$$\lim_{r\to 0} \left\{ \int_{S_r} \left(K^2|W|^2 + |W_r|^2 \right) ds + iK \int_{S_r} \left(\overline{W}W_r - W\overline{W}_r \right) ds \right\} = 0. \tag{13}$$

If we prove

$$\int_{S_r} \left(\overline{W}W_r - W\overline{W}_r \right) ds = 0, \tag{14}$$

for $r > a$, then (13) and Lemma 1 imply that $W = 0$ in $\mathbb{R}^n \setminus B_a$, and $W = 0$ in D' from Lemma 2.

To derive (14), take in (11), with $f = 0$, the test function $\overline{w} = \overline{W}h((|x| - r_0)/\epsilon) := \overline{W}h_\epsilon$, $r_0 > a$, where $h \in C^\infty(\mathbb{R})$, $h \geq 0$, $h(r) = 0$ if $r < 1/2$ and $h(r) = 1$ if $r > 3/2$.

Subtract from expression (11) its complex conjugate. The real-valuedness of q, σ, and K, and the assumptions on a_{ij}, imply

$$\int_{D'} \left(W_j \overline{W} - \overline{W}_j W\right)(h_\epsilon)_j dx = 0.$$

We have $(h_\epsilon)_j = h'((|x| - r_0)/\epsilon)(x_j/(\epsilon|x|))$, $(1/\epsilon)h'((|x| - r_0)/\epsilon) \to \delta(|x| - r_0)$, and $W \in C^\infty(|x| > a)$. Therefore we can take $\epsilon \to 0$ and get (14).

Note that in this argument assumption (A) is not used. Theorem 1 is proved. ∎

The following lemma will be used in Section 3.

LEMMA 3. *If W is a solution of the equation $LW = 0$ in $\mathbb{R}^n$, which satisfies radiation condition* (12), *then* $W = 0$.

Proof. As in the proof of Theorem 1, if suffices to prove (14). We have $\overline{W}LW - WL\overline{W} = 0$. Integrate over B_r and use Green's formula to get

$$0 = \int_{B_r} \left(L[\overline{W}, W] - L[W, \overline{W}]\right)dx + \int_{S_r} \left(\overline{W}W_r - W\overline{W}_r\right)ds.$$

Assumptions (6) and (8) imply that the first integral vanishes, we get (14), and Lemma 1 implies $W = 0$. ∎

3. EXISTENCE

THEOREM 2. *If assumptions* (A) *and* (6)–(9) *hold, then there exists a solution to the scattering problem and this solution is unique.*

As in Theorem 1, it is sufficient to prove the existence of a function $W \in \mathscr{H}$ such that (11) and (12) hold with f supported in the annulus $r_0 < |x| < r_0 + 1$, $r_0 > a$. We give the argument for the Neumann condition ($\sigma = 0$), the case of the Robin condition is treated similarly.

The idea of the proof is to reduce the scattering problem to a Fredholm-type equation without using integral equations, and to derive the existence of its solution from the uniqueness of the solution, which is a consequence of Theorem 1. We first prove an auxiliary result stated in Proposition 1, then describe the above reduction, and then complete the proof of Theorem 2.

Let us pass to the auxiliary result. Consider the problem

$$LV = h \text{ in } \mathbb{R}^n, \qquad V \text{ satisfies the radiation condition } (4). \tag{15}$$

Denote $H^0 := L_0^2(D_R)$, $R > r_0 + 1$, the set of $L^2(D_R)$ functions vanishing near S_R, and assume that $\operatorname{supp} h \subset D_R$.

PROPOSITION 1. *Given an arbitrary $h \in H^0$ there exists a unique solution V of* (15) *such that $V \in H^2_{loc}$ and $\|V\|_b < \infty$, $b > 1$, where*

$$\|V\|_b^2 := \int_{\mathbb{R}_n} \frac{|V|^2}{(1 + |x|)^b} dx.$$

Proof of Proposition 1. From (6)–(8) it follows that L is a symmetric, semibounded from below, operator in L^2 with domain H_0^2, the set of $H^2(\mathbb{R}^n)$ functions vanishing near infinity. We can take its Friedrichs' extension, also called L, defined on the dense subset of the domain of the quadratic form associated to $L + C > 0$, which is H^1. From the self adjointness of L it follows that for $\epsilon > 0$ the function $V_\epsilon := (L + i\epsilon)^{-1} h$ is well defined. The following lemma yields the conclusions of Proposition 1.

LEMMA 4. *We have*

$$V_\epsilon \to V \qquad \text{as } \epsilon \to 0 \text{ in } H^2_{loc} \text{ and in } \|\ \|_b, \tag{16}$$

and

$$V \text{ solves } (15). \tag{17}$$

We prove Lemma 4 in two steps.

Step 1. Under the assumption

$$\sup_{0 < \epsilon < \epsilon_0} \|V_\epsilon\|_b < \infty, \tag{18}$$

the assertions (16) and (17) hold.

Step 2. Under the hypothesis of Proposition 1, inequality (18) holds.

If (18) holds, then for any ball B_R, $\|V_\epsilon\|_{L^2(B_R)} < C(R)$, and therefore V_ϵ converges weakly in $L^2(B_R)$ to some function V. We have $LV_\epsilon = -i\epsilon V_\epsilon + h$, $\|LV_\epsilon\|_{L^2(B_R)} < C(R, h)$, and from the interior elliptic estimates it follows that $\|V_\epsilon\|_{H^2(B'_R)} < C(R, R', h)$; $R' < R$. Therefore, for any $R < a$, $\|V_\epsilon\|_{L^2(B_R)} < C(R)$, $V_\epsilon \to V$ in $H^s(B_R)$, $s < 2$. Using again (15), we conclude that LV_ϵ converges in $L^2(B_R)$, and from the interior elliptic estimates we know that

$$\|V_\epsilon - V_{\epsilon'}\|_{H^2(B'_R)} \le C\big(\|LV_\epsilon - LV_{\epsilon'}\|_{L^2(B_R)} + \|V_\epsilon - V_{\epsilon'}\|_{L^2(B_R)}\big).$$

It follows that $V_\epsilon \to V$ strongly in H^2_{loc}. We can pass to the limit in the equation and get $LV = h$.

We now prove convergence in the norm $\| \ \|_b$ and verify the radiation condition for V. In B'_R, $R > a$, V_ϵ satisfies the equation $(\Delta + K^2 + i\epsilon)V_\epsilon = 0$. Therefore, Green's formula yields

$$V_\epsilon(x) = \int_{S_R} \left[V_\epsilon(s)\big(g_\epsilon(x-s)\big)_r - (V_\epsilon)_r(s) g_\epsilon(x-s) \right] ds, \qquad x \in B'_R, \tag{19}$$

where g_ϵ is the fundamental solution

$$g_\epsilon(x) := \frac{(i\epsilon + K^2)^{(n-2)/2}}{|x|^{(n-2)/2}\omega_n} H^{(1)}_{(n-2)/2}\Big((i\epsilon + K^2)^{1/2}|x| \Big),$$

and we take $\Im(i\epsilon + K^2)^{1/2} > 0$.

From (19) it follows that

$$|V_\epsilon(x)| \le \frac{c}{|x|^{(n-1)/2}}, \qquad \text{if } |x| > R_0, 0 < \epsilon < \epsilon_0. \tag{20}$$

Since $V_\epsilon \to V$ strongly in H^2_{loc}, we can pass to the limit in (19) and get

$$V(x) = \int_{S_R} \left[V(s)\big(g_0(x-s)\big)_r - (V)_r(s) g_0(x-s) \right] ds, \qquad x \in B'_R.$$

From this representation of V it follows that V satisfies the radiation condition.

Finally, from estimate (20) and the pointwise convergence $V_\epsilon \to V$ it follows that $\|V_\epsilon - V\|_b \to 0$ and Step 1 is completed.

Let us pass to Step 2.

Arguing by contradiction, assume the existence of sequences $\epsilon_k \to 0$, $V_k := V_{\epsilon_k}$, such that $\|V_k\|_b \to \infty$. Since $V_\epsilon = (L + i\epsilon)^{-1}h$ and $\|V_\epsilon\|_b \le (1/\epsilon)\|h\|_{L^2}$, we can define $W_k := V_k/\|V_k\|_b$. Then

$$(L + i\epsilon)W_k = \frac{h}{\|V_k\|_b} := h_k \qquad \text{and} \qquad \|h_k\|_{L^2(B_R)} \to 0. \tag{21}$$

Since $\|W_k\|_b = 1$, under condition (21) we can repeat the arguments in Step 1 and prove that $W_k \to W$ in H^2_{loc}, W satisfies the radiation condition, $\|W_k - W\|_b \to 0$, and $LW = 0$. From the uniqueness Lemma 3 it follows that $W = 0$. This contradicts the relations $\|W_k - W\|_b \to 0$ and $\|W_k\|_b = 1$. Proposition 1 is proved. ∎

Let us now describe the reduction of the scattering problem in the form (11)–(12) to a Fredholm-type equation ([3, p. 36]). Set

$$W := V - \eta Z, \tag{22}$$

where V solves (15) and η is a C^∞ function which equals 1 near S and 0 outside D_R. In particular, $\eta = 0$ on S_R. The function W solves problem (11)–(12) if Z solves the problem

$$-f = h - L(\eta Z) \text{ in } D_R, \qquad \Gamma(Z) = \Gamma(V), \qquad Z = 0 \text{ on } S_R, \tag{23}$$

where we have used the strong formulation of the problem for convenience of the reader and with the understanding that (23) is understood in the weak sense, similarly to (11). Note that near infinity W, defined in (22), equals V and, by the definition of V, satisfies the radiation condition (12). There are many Z which solve (23) (since h in (23) is arbitrary). Let us fix a unique Z as the solution to the problem

$$LZ = iZ \text{ in } D_R, \qquad \Gamma(Z) = \Gamma(V) \text{ on } S, \qquad Z = 0 \text{ on } S_R, \tag{24}$$

where again the weak formulation via the integral identity is understood. Clearly the solution to (24) is a linear operator on h. Define $Bh := L(\eta Z)$. Then Eq. (23) can be written as

$$h - Bh = -f. \tag{25}$$

The operator B in (25) is compact in H^0. Indeed, $L(\eta Z) = \eta LZ + QZ$, where $QZ := L(\eta Z) - \eta LZ$ contains not higher than the first derivatives of Z and $LZ = iZ$ by (24). The map $h \to V \to Z \to QZ$ is the map $H^0 \to H^1$, as follows from the known estimates for the solutions of second order elliptic equations in bounded domains (recall that QZ vanishes near non-smooth boundary S because $\eta = 1$ near S). By assumption (A1), the embedding $H^0 \to H^1$ is compact, so B is compact in H^0. If the Robin boundary condition is used, then we use assumption (A2) also.

Suppose that h solves (25), V solves (15), and W is defined by (22). Then W solves (11)–(12), as we checked above. Therefore, the scattering problem in the form (11)–(12) is reduced to Eq. (25) with compact operator B in H^0. The proof of Theorem 2 will be completed as soon as we show that the homogeneous version of Eq. (25) has only the trivial solution. Let us show this and thus complete the proof of Theorem 2.

Assume that h solves (25) with $f = 0$. Then W, defined by (22), solves (11)–(12) with $f = 0$. By Theorem 1, we get $W = 0$ in D'. Thus, $V = \eta Z$ in D_R, and $V = Z$ on S. This and the first boundary condition (24) imply that V and Z have the same Cauchy data on S. Therefore, the function Z, extended into D so that $Z = V$ in D, solves the problem

$$LZ = iZ \text{ in } D_R, \qquad LZ = 0 \text{ in } D, \qquad Z = 0 \text{ on } S_R, \tag{26}$$

where the equation $LZ = 0$ in D follows from (15) and the fact that $h = 0$ in D. Since L is symmetric, it follows from (26) that $Z = 0$ in B_R. Therefore $V = 0$ in D_R, and $h = LV = 0$ in D_R. Theorem 2 is proved. ∎

Remark 1. We can relax condition (8): allow for $q \in L^p_{loc}$, $p > n/2$, and allow $q = o(|x|^{-1})$ as $|x| \to \infty$. For such q, Lemma 1 remains valid if the Laplacian in this lemma is replaced by $\Delta - q(x)$ with the above decay property at infinity (Kato's theorem).

REFERENCES

1. C. Kenig, ''Harmonic Analysis Techniques for Second Order Elliptic Boundary Problems,'' Amer. Math. Soc., Providence, Rhode Island, 1994.
2. V. Mazja, ''Sobolev Spaces,'' Springer-Verlag, New York, 1985.
3. A. G. Ramm, ''Scattering by Obstacles,'' Reidel, Dordrecht, 1986.
4. A. G. Ramm, Uniqueness theorems for inverse obstacle scattering problems in Lipschitz domains, *Appl. Anal.*, **59**, (1995), 377–383.
5. C. Wilcox, *Scattering theory for the D'Alembert equation in exterior domains*, in ''Lecture Notes in Math.,'' Vol. 442, Springer-Verlag, New York, 1975.